# CS205 Homework #8 Review Session Notes

## Properties of First Order ODEs

Most properties of first order scalar ODEs ($y' = \lambda y$) extend naturally to their vector-valued counterparts ($\mathbf{y}' = \mathbf{A}\mathbf{y}$).

1. **Existence and Uniqueness**. Consider the following scalar and vector-valued initial value problems:

$$y' = \lambda y, \quad y(t_0) = y_0$$
$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{y}_0$$

   In both cases, a unique solution exists.

2. **Well-Posedness**. We say that an initial value problem is strictly well-posed if its analytic solution decays to zero as $t \to \infty$, regardless of the initial value condition itself. For the scalar ODE $y' = \lambda y$ the necessary and sufficient condition for well-posedness is $\lambda < 0$. For the vector-valued case, the corresponding condition is that $Re\{\lambda\} < 0$ for any eigenvalue $\lambda$ of $\mathbf{A}$.

   There is also a "relaxed" definition for well-posedness, which is much less common. The relaxed definition requires only that the analytic solution to an initial value problem remains *bounded*, regardless of the initial conditions. The necessary and sufficient condition for the scalar case is $\lambda \leq 0$. The vector-valued case, however, is more complex. If $\mathbf{A}$ is *diagonalizable* (that is, has a complete set of eigenvectors) then the corresponding condition is $Re\{\lambda\} \leq 0$, for all of its eigenvalues. If, however, the matrix $\mathbf{A}$ is *defective*, we also need that $Re\{\lambda\} < 0$ for any eigenvalue $\lambda$ with geometric multiplicity less than its algebraic multiplicity. This subtlety contributes to the fact that this relaxed definition of well-posedness is only rarely used in practice.

3. **Stability**. The concept of stability refers to a particular numerical method for solving a differential equation rather than to the differential equation itself. A stable method has the property that, when applied to a well-posed ODE, the *successive iterates* (computed solution values) decay to zero as the number of iterations increases to infinity. The relaxed concept of neutral stability merely requires those iterates to remain bounded, but we will not examine this branch of the theory in depth.

   Consider the the special case of a numerical method that can be written as a one-point linear recursion:

$$y_{k+1} = my_k + b$$
$$\mathbf{y}_{k+1} = \mathbf{M}\mathbf{y}_k + \mathbf{b}$$

   Here we have that the condition for stability is $|m| < 1$ for the scalar case and $\rho(\mathbf{M}) < 1$ for the vector valued case, where $\rho(\mathbf{M}) = \|\lambda_{\max}\|$ is the *spectral radius* of $\mathbf{M}$, or the magnitude of its largest eigenvalue (which may or may not be complex).

Several important numerical methods for solving ODE's can be written as one-point linear recursions. For example, forward and backward Euler applied to the scalar equation $y' = \lambda y$ respectively yield:

$$y_{k+1} = (1 + \lambda \Delta t) y_k$$

$$y_{k+1} = \frac{1}{1 - \lambda \Delta t} y_k$$

For the vector-valued case, we have:

$$\mathbf{y}_{k+1} = (\mathbf{I} + \Delta t \mathbf{A}) \mathbf{y}_k$$

$$\mathbf{y}_{k+1} = (\mathbf{I} - \Delta t \mathbf{A})^{-1} \mathbf{y}_k$$