CS205 Review Session #7 Notes

Useful Decompositions

Consider an $n \times n$ matrix $\mathbf{A}$ that is symmetric positive semi-definite and has a nullspace of dimension $p < n$. We wish to factor $\mathbf{A}$ into $\mathbf{M}\tilde{\mathbf{A}}\mathbf{M}$ where the columns of the $n \times (p-n)$ matrix $\mathbf{M}$ form an orthonormal basis for col($\mathbf{A}$) = row($\mathbf{A}$). It may help to first consider a concrete example:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T$$

This particular choice for $\tilde{\mathbf{A}}$ and $\mathbf{M}$ is not necessarily unique. We can also write:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

One way to construct this factorization in general is to compute the spectral decomposition $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$ of $\mathbf{A}$ and trim of some of the unnecessary columns of $\mathbf{Q}$.

We can show, however, that no matter what method we use to determine $\mathbf{A} = \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T$, if the columns of $\mathbf{M}$ form an orthonormal basis for the column space of $\mathbf{A}$, $\tilde{\mathbf{A}}$ must have several desirable properties. In particular:

1. $\tilde{\mathbf{A}}$ has full rank. To see why this is so, let $\tilde{\mathbf{x}} \in \mathbb{R}^{n-p}$ be an arbitrary vector. Then $\mathbf{M}\tilde{\mathbf{x}} \in$ col($\mathbf{A}$) = row($\mathbf{A}$). This means that $\mathbf{A}(\mathbf{M}\tilde{\mathbf{x}}) = \mathbf{0} \iff \tilde{\mathbf{x}} = \mathbf{0}$ since the nullspace of a matrix is the orthogonal complement of its row space. This implies that, for general $\tilde{\mathbf{x}}$:

   $$\mathbf{A}\mathbf{M}\tilde{\mathbf{x}} \neq \mathbf{0} \Rightarrow \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T\mathbf{M}\tilde{\mathbf{x}} \neq \mathbf{0} \Rightarrow \mathbf{M}\tilde{\mathbf{A}}\tilde{\mathbf{x}} \neq \mathbf{0} \Rightarrow \tilde{\mathbf{A}}\tilde{\mathbf{x}} \neq \mathbf{0}$$

   Therefore $\tilde{\mathbf{A}}$ has only the trivial nullspace, and thus has full rank.

2. $\tilde{\mathbf{A}}$ is symmetric. $\mathbf{A} = \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T \Rightarrow \tilde{\mathbf{A}} = \mathbf{M}^T\mathbf{A}\mathbf{M}$, which is symmetric since $\mathbf{A}$ is.

3. $\tilde{\mathbf{A}}$ is positive definite. $\tilde{\mathbf{A}}$ must be positive semi-definite since $\tilde{\mathbf{x}}^T\tilde{\mathbf{A}}\tilde{\mathbf{x}} = (\mathbf{M}\tilde{\mathbf{x}})^T\mathbf{A}(\mathbf{M}\tilde{\mathbf{x}}) \geq 0$. However, $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ in general if and only if $\mathbf{x} \in$ NS($\mathbf{A}$). Since we chose $\mathbf{M}$ to have columns that form an orthonormal basis for col($\mathbf{A}$), $\mathbf{M}\tilde{\mathbf{x}} \in$ col($\mathbf{A}$) = NS($\mathbf{A}$)$^\perp$ and so $\tilde{\mathbf{x}}^T\tilde{\mathbf{A}}\tilde{\mathbf{x}} > 0$.

4. The eigenvalues of $\tilde{\mathbf{A}}$ are exactly the non-zero eigenvalues of $\mathbf{A}$. To see that this is the case, consider an eigenvector $\tilde{\mathbf{x}}$ of $\tilde{\mathbf{A}}$ and its associated eigenvalue $\tilde{\lambda}$:

   $$\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \lambda\tilde{\mathbf{x}} \Rightarrow \tilde{\mathbf{A}}\mathbf{M}^T\mathbf{M}\tilde{\mathbf{x}} = \lambda\tilde{\mathbf{x}} \Rightarrow \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T\mathbf{M}\tilde{\mathbf{x}} = \lambda\mathbf{M}\tilde{\mathbf{x}} \Rightarrow \mathbf{A}(\mathbf{M}\tilde{\mathbf{x}}) = \lambda(\mathbf{M}\tilde{\mathbf{x}})$$

More Subspace Decompositions

Consider any of the (possibly infinite) solutions to the matrix equation $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is symmetric. We recall that we can express each such solution as a sum $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_n$ where $\mathbf{x}_c \in \text{col}(\mathbf{A}) = \text{row}(\mathbf{A})$ and $\mathbf{x}_n \in \text{NS}(\mathbf{A})$.

Now, consider a solution that lies entirely within the column space of $\mathbf{A}$, that is, a solution of the form $\mathbf{x}_c$ where $\mathbf{Ax}_c = \mathbf{b}$.

(a) $\mathbf{x}_r$ is unique. To see that this is the case, assume that $\mathbf{Ax}_{r1} = \mathbf{b}$ and $\mathbf{Ax}_{r2} = \mathbf{b}$ where $\mathbf{x}_{r1}, \mathbf{x}_{r2} \in \text{col}(\mathbf{A})$. Clearly $(\mathbf{x}_{r1} - \mathbf{x}_{r2}) \in \text{col}(\mathbf{A})$, but we also have that $\mathbf{A}(\mathbf{x}_{r1} - \mathbf{x}_{r2}) = \mathbf{Ax}_{r1} - \mathbf{Ax}_{r2} = \mathbf{b} - \mathbf{b} = \mathbf{0}$, which implies that $(\mathbf{x}_{r1} - \mathbf{x}_{r2}) \in \text{NS}(\mathbf{A})$, yielding a contradiction.

(b) $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ is also a solution to $\mathbf{Ax} = \mathbf{b}$ for *any* $\mathbf{x}_n \in \text{NS}(\mathbf{A})$. To see that this is true, consider $\mathbf{A}(\mathbf{x}_r + \mathbf{x}_n) = \mathbf{Ax}_r + \mathbf{Ax}_n = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Particular Polynomials

The generic form of a quadratic polynomial is:

$$p(x) = c_2 x^2 + c_1 x + c_0$$

However, deriving a system of equations for the constraints $p'(a) = f'(a), p'(b) = f'(b), p(\frac{a+b}{2}) = f(\frac{a+b}{2})$ for some generic $f(x)$ is quite cumbersome if we express $p(x)$ in this manner.

Instead, we may substitute a different quadratic polynomial $g(x)$ that benefits from the symmetry of the given constraints:

$$g(x) = c_2 \left(x - \frac{a+b}{2}\right)^2 + c_1 \left(x - \frac{a+b}{2}\right) + c_0$$

Remember also that for a non-composite rule, the degree is given as the *maximum* degree of a polynomial that is exactly integrable using the particular rule. Note that it is enough to test exact integrability of the simple monomials $(1, x, x^2, x^3, \ldots)$ since, by linearity, any linear combination of exactly integrable functions will be exactly integrable as well.