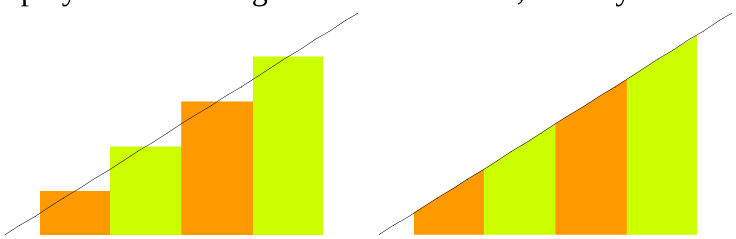
<u>CS205 – Class 14</u>

Covered in class: 1, 3, 4
Readings: 7.4, 8.1 to 8.3

Quadrature

1. **<u>Numerical quadrature</u>** approximate $I = \int_a^b f(x)dx$ for a given f
   a. These f's might be arbitrarily difficult to compute and only available by running a program.
   b. General approach : Subdivide $[a,b]$ into n intervals $[x_i, x_{i+1}]$ with $x_0 = a$ and $x_n = b$ and consider each subinterval separately
2. **<u>Newton-Cotes quadrature</u>** for each subinterval $[x_i, x_{i+1}]$, choose n equally spaced points and use k-1 degree polynomial interpolation to approximate the integral
   a. Exact on polynomials of degree n-1 when n is even, as expected
   b. Exact on polynomials of degree n when n is odd, from symmetric cancellation
      i. 
   c. **<u>local accuracy</u>** an exact method on k degree polynomials has a local error that scales like $O(h^{k+2})$ in each subinterval where h is the length of the subinterval
   d. **<u>global accuracy</u>** since there are $O(1/h)$ subintervals, the total error scales like $O(h^{k+1})$
      i. doubling the number of subintervals, sends $h \to h/2$, and reduces the error by $(1/2)^{k+1}$
      ii. order of accuracy is k+1
   e. **<u>Midpoint rule</u>** $M = \sum (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right)$
      i. n=1 point, piecewise constant, exact for piecewise linear functions, 2nd order accurate
   f. **<u>Trapezoidal rule</u>** $T = \sum \left(\frac{x_{i+1} - x_i}{2}\right)\left(f(x_i) + f(x_{i+1})\right)$
      i. n=2 points, piecewise linear, exact for piecewise linear functions, 2nd order accurate
   g. **<u>Simpson's rule</u>** $S = \sum \left(\frac{x_{i+1} - x_i}{6}\right)\left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right)$

i. n=3 points, piecewise quadratic, exact for piecewise cubic functions, 4^{th} order accurate.

ii. Notice the huge jump from Trapezoidal rule. Basically Trapezoidal is where it should be and midpoint actually got promoted and here Simpson's got promoted.

h. Careful not to evaluate endpoints twice - factor out $h = x_{i+1} - x_i$ when possible

i. Typically in most problems we don't get the extra order for free. Typically lots of work is spent just trying to get the simplest first order accurate methods to work. Then there are a bunch of researchers trying to extend those methods, and there is very few people working on getting up to typically third order accuracy.

3. **Gaussian Quadrature** – for each subinterval $[x_i, x_{i+1}]$, use k specially spaced points to obtain a method that is exact on 2k-1 degree polynomials and thus has an order of accuracy of 2k

a. $G = \sum \left(\frac{x_{i+1} - x_i}{2} \right) \left(f\left(\frac{x_i + x_{i+1}}{2} - \frac{x_{i+1} - x_i}{2\sqrt{3}} \right) + f\left(\frac{x_i + x_{i+1}}{2} + \frac{x_{i+1} - x_i}{2\sqrt{3}} \right) \right)$

i. 2 points, piecewise cubic, exact for piecewise cubic functions, 4^{th} order accurate