CS 205a Fall 2011 Midterm 2

Please write your name and netid on the top right of the first page. The exam is closed book and no calculators are allowed. You have 1 hour and 15 minutes to complete the exam.

Multiple Choice (4 x 1 pt each)

For each of the following questions, circle **ALL** answers that are correct.

1. Let A be a symmetric positive definite $n \times n$ matrix that you attempt to solve using the Conjugate Gradient method. If A has 3 distinct eigenvalues, what is the maximum number of steps (in theory) that the CG solver needs to converge?
   (a) n
   (b) 3
   (c) $n - 3$
   (d) None of the above is correct.

2. Recall the Newmark method:

$$x^{n+1} = x^n + \Delta t v^n + \frac{\Delta t^2}{2}\left[(1-2\beta)\alpha^n + 2\beta\alpha^{n+1}\right]$$

$$v^{n+1} = v^n + \Delta t\left[(1-\gamma)\alpha^n + \gamma\alpha^{n+1}\right]$$

   Which of the following methods is not equivalent for some choice of α and β parameters?
   (a) Constant Acceleration
   (b) 3^{rd}-order accurate Runge-Kutta.
   (c) Central Differencing
   (d) Trapezoidal Rule

3. In general, which of the following methods are the most suitable for solving heat equations?
   (a) 2^{nd}-order Runge Kutta.
   (b) Forward Euler.
   (c) Backward Euler.
   (d) 4^{th}-order Runge Kutta.

4. The Monte Carlo method
   (a) is widely used for problems in lower dimensions
   (b) has an error proportional to $1/\sqrt{n}$
   (c) is widely used for problems in higher dimensions
   (d) is deterministic

Conjugate Gradient (10 pts)

1. Apply the Conjugate Gradient method to find the solution of the following linear system
   $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
   Use the initial guess $\vec{x}_0 = [0, 0]^T$. (5 pts)

2. Let the CG iteration be applied to solve a linear system $AX = b$ with A as a symmetric positive definite matrix. $\langle s_0, s_1, \cdots, s_k, \cdots, s_n \rangle$ is the sequence of search directions generated in the iteration. Assume the initial error can be expressed as $e_0 = \sum_{j=0}^{n} c_j s_j$. First derive the relations between step size of CG at k^{th} step, i.e. α_k, and the coefficient c_k; Second, derive the expression of the error after k iterations e_k. (5 pts)

Optimization (6 pts)

Minimize $f(x) = 5x_1 + 3x_2^2 + x_3^2 + 2x_4$ subject to the constraints $x_1 + x_2 = x_3$ and $x_2 - x_4 = 2$ using Lagrange multipliers. Show your work, including a problem of the form $Ax = b$.

Ordinary Differential Equations (10 pts)

The following scheme has been proposed for solving $y' = f(y)$:

$$y^* = y_n + \frac{h}{2} f(y_n)$$
$$y_{n+1} = y_n + h f(y^*)$$

with h being the time step.

1. Applying this method to the model problem $y' = \lambda y$, what is the maximum step size h for λ being negative real in order to have a stable solution? (4 pts)

2. Still applying this method to the model problem $y' = \lambda y$, what is the order of global accuracy of this method? (4 pts)

3. Convert the sixth-order non-linear ODE $y^{(6)} = 3y^{(4)} + 4y^{'''} * y^{''} - (y^{'} + y) * y^{(5)}$ into a system of first-order equations. Show your variable assignments. (2 pts)

Interpolation (10 pts)

1. State the main disadvantage of the monomial basis functions. (1 pt)

2. Given the set of sample points (-2,12), (-1,3), (0,10) and (3,7), construct an interpolating polynomial using monomial basis functions. (4 pts)

3. Given the same set of sample points (-2,12), (-1,3), (0,10) and (3,7), construct an interpolating polynomial using Newton interpolation. (3 pts)

4. Suppose the function that these points are sampled from is $O(x^3)$. What does that imply about your answers in parts 2 and 3? (1 pt)

5. What are the advantages and disadvantages of Lagrange interpolation? (1 pt)