CS 205a Fall 2011 Midterm 1

Please write your name on the top right of this page. The exam is closed book/closed notes and no calculators are allowed. You have 1 hour and 15 minutes to complete the exam.

Multiple Choice (4 x 1 pt each)

For each of the following questions, circle all answers which are correct. You must circle **ALL** of the answers for a given question correctly to receive credit.

1. Which of the following matrices are always diagonalizable (i.e. the matrices that always have a full set of eigenvectors)?
   (a) Symmetric matrices
   (b) Orthogonal matrices
   (c) Householder matrices
   (d) Upper triangular matrices

2. Which of the following computations have operation counts of $O(n^3)$? Consider the cases for general $n \times n$ matrices.
   (a) QR factorization with the Gram-Schmidt method
   (b) LU decomposition
   (c) Back substitution step in solving $A\vec{x} = \vec{b}$
   (d) Cholesky factorization

3. $\|\vec{x}\|_\infty$ is best for evaluating the numerical solution to a system of equations $A\vec{x} = \vec{b}$ because:
   (a) It is often the only obtainable norm
   (b) It illustrates the worst error for a solution to an individual equation
   (c) It provides information about the total error in the solution
   (d) It is always cheapest to compute

4. A Householder matrix $H = I - 2\frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}$
   (a) $\forall \vec{x} \in \mathbb{R}^n, \|\vec{x}\|_2 = \|H\vec{x}\|_2$
   (b) has eigenvalues equal to 1 with multiplicity n
   (c) is a projection matrix onto the hyperplane orthogonal to $\vec{v}$
   (d) has a determinant of -1 (i.e. $det(H) = -1$)

Eigenvalues and Eigenvectors (10 pts)

Given a 2×2 matrix $A = \begin{bmatrix} 7 & 2 \\ 3 & 2 \end{bmatrix}$, answer the following:

1. What are the eigenvalues of A? (3 pts)

2. For each eigenvalue of A, find the corresponding eigenvectors. (3 pts)

3. If you were to perform power method iterations on A with $\vec{x_0} = \begin{pmatrix} 3 & 2 \end{pmatrix}^T$ as the starting vector, to what eigenvector will the power method converge? Why? (1 pt)

4. Recall that if $A\vec{x} = \lambda\vec{x}$, one can form the Rayleigh Quotient. Use $\vec{x_0} = \begin{pmatrix} 3 & 2 \end{pmatrix}^T$ as an approximate eigenvector, and compute the corresponding approximate eigenvalue using the Rayleigh Quotient. You may round to an integer in the final calculation. To which exact eigenvector, calculated in (2), does this approximate eigenvalue correspond? In general, how can the Rayleigh Quotient be used to accelerate the power method? (3 pts)

Optimization and Nonlinear Equations (7 pts)

1. Rank the following methods from slowest convergence rate to fastest convergence rate: Newton's Method, Secant Method, Bisection. Which of these is the most robust? Devise a method to maintain the fastest convergence rate while preserving robustness. (3 pts)

2. Suppose you use an iterative method to solve $f(x) = 0$ for a root that is ill-conditioned, and you need to choose a convergence test. Would it be better to terminate the iteration when you find x_k for which $|f(x_k)|$ is small, or when $|x_k - x_{k-1}|$ is small? Why? (3 pts)

3. Draw a 2-dimensional example of convergence of the steepest descent method. Clearly indicate the angles between search direction and contours of the objective function. (1 pt)

Least Squares (9 pts)

1. Prove that the method of normal equations minimizes the residual. (3 pts)

2. On the computer, why do we solve the least squares problem $\min_{\vec{x}} \|\vec{b} - A\vec{x}\|$, even when A is invertible? (2 pts)

3. In what case(s) do we want to avoid solving a linear system on the computer using the normal equations, and why? (1 pt)

4. Use the modified Gram-Schmidt method to find the QR decomposition of $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. (3 pts)

Singular Value Decomposition (10 pts)

1. Write down the SVD for the following matrices (no proof is necessary): (3 pts)

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}:$$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}:$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}:$$

2. Prove that if A is a real matrix and $A = A^T$, then the singular values of A are the absolute values of the eigenvalues of A. (1 pt)

3. Given the SVD of A as $A = U\Sigma V^T$, what space do the columns of V corresponding to zero singular values span? Prove your statement. (3 pts)

4. Let A be an $m \times n$ real matrix. Consider a symmetric matrix eigenvalue problem:

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \vec{x} = \lambda \vec{x}$$

Show that if λ satisfies this relation, then $|\lambda|$ is a singular value of A. (3 pts)