

CS205 Homework #8 Solutions

Problem 1

Give a criterion for the well-posedness of the k th order, scalar, homogeneous, constant-coefficient ODE

$$u^{(k)} + c_{k-1}u^{(k-1)} + \cdots + c_1u' + c_0u = 0$$

(Hint: Transform to a first-order system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and observe \mathbf{A} is a matrix we've encountered previously in homework 3 problem 2)

Solution

Transforming the differential equation into a system of first order equations yields:

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{k-1}' \\ u_k' \end{bmatrix} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_k \\ -\sum_{i=1}^k c_{i-1}u_i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{k-2} & -c_{k-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}$$

The matrix is a companion matrix as we saw in homework 3. Recall its characteristic polynomial is $p(\lambda) = c_0 + c_1\lambda + \cdots + c_{k-1}\lambda^{k-1} + \lambda^k$. The eigenvalues of the matrix will be the roots of this polynomial. Thus, if the real parts of the roots are less than zero it is well-posed. If they are all not strictly less than zero then it is asymptotically stable. If any real part is positive then it is ill-posed. As an aside, if any of the roots are pure imaginary, then it automatically ill-posed.

Problem 2

Consider the system of linear ODE's

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

1. Consider the initial value problem with the above ode and the initial values

$$y_1(0) = y_2(0) = 1$$

Show that the analytic solution to this initial value problem is

$$y_1(t) = y_2(t) = e^{-t}$$

2. If we use an integration method (such as Forward/Backward Euler, or trapezoidal rule) to compute the solution to this ODE numerically, will we get the same asymptotic behavior as the analytic solution as $t \rightarrow \infty$?

Solution

1. The eigenvalues of \mathbf{A} are 3 and -1. The corresponding eigenvectors are $\begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}$. Since \mathbf{A} is symmetric we have $\mathbf{A} = \mathbf{P}^T \mathbf{\Lambda} \mathbf{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. We can write our differential equation system as $\mathbf{y}_t = \mathbf{A} \mathbf{y} = \mathbf{A} = \mathbf{P}^T \mathbf{\Lambda} \mathbf{P} \mathbf{y} \Rightarrow \mathbf{P} \mathbf{y}_t = \mathbf{\Lambda} \mathbf{P} \mathbf{y}$. Substituting $\mathbf{u} = \mathbf{P} \mathbf{y}$ we have $\mathbf{u}_t = \mathbf{\Lambda} \mathbf{u}$. The solution to this system is trivially $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix}$. Transforming our initial condition yields the constant $\mathbf{u}(0) = \mathbf{P}^T \mathbf{y}(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$. Thus we have $\mathbf{u} = \begin{bmatrix} 0 & \sqrt{2} e^{-t} \end{bmatrix}^T$. Substituting back we get

$$\mathbf{y} = \mathbf{P}^t \mathbf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

2. The initial condition happens to be an eigenvector of the negative (-1) eigenvalue of the system. This causes the analytic solution to decay to zero as would be expected of a stable system. Nevertheless, during the solution process, if a tiny amount of error infiltrates the solution, this error is going to have a component along the direction of the eigenvector corresponding to the positive eigenvalue (3). At that point even an

optimal numerical solver (one that computes the exact analytical solution given that initial value) will amplify this component of the solution exponentially, causing the whole thing to blow up.

Problem 3

Consider the equation of motion for a simple, damped, 1D oscillator (a zero rest length spring in 1D with damping)

$$F(x, v) = ma = -bv - kx$$

where k is the spring constant, b the (constant) damping coefficient, $v = x_t$ the velocity and $a = v_t = x_{tt}$ the acceleration.

1. Show that this 2nd order ODE is equivalent to the 1st order linear system of ODEs

$$\begin{pmatrix} x \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

2. Assume that we are using Forward Euler to solve this system numerically, with a timestep equal to Δt . If $\lambda_1, \lambda_2 \in \mathbb{C}$ are the complex eigenvalues of the matrix

$$\begin{pmatrix} 1 & \Delta t \\ -\frac{k\Delta t}{m} & 1 - \frac{b\Delta t}{m} \end{pmatrix}$$

show that the condition for stability is $\|\lambda_1\| < 1$ and $\|\lambda_2\| < 1$

3. Show that if $b^2 < 4km$ (such spring systems are referred to as *under-damped*), then the eigenvalues of the matrix above are given as

$$\lambda_{1,2} = 1 - \frac{b\Delta t}{2m} \pm i \frac{\Delta t}{2m} \sqrt{4km - b^2}$$

4. Show that if $b^2 < 4km$ the condition for stability is $\Delta t < b/k$.

Solution

1. We have $mv_t = ma = -bv - kx \Rightarrow v_t = -\frac{b}{m}v - \frac{k}{m}x$. Therefore

$$\left. \begin{array}{l} x_t = v \\ v_t = -\frac{b}{m}v - \frac{k}{m}x \end{array} \right\} \Rightarrow \begin{pmatrix} x \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

2. Forward Euler can be written as

$$\begin{aligned} \begin{pmatrix} x \\ v \end{pmatrix}^{n+1} &= \begin{pmatrix} x \\ v \end{pmatrix}^n + \Delta t \begin{pmatrix} x_t \\ v_t \end{pmatrix}^n \Rightarrow \\ \begin{pmatrix} x \\ v \end{pmatrix}^{n+1} &= \begin{pmatrix} x \\ v \end{pmatrix}^n + \Delta t \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}^n \Rightarrow \\ \begin{pmatrix} x \\ v \end{pmatrix}^{n+1} &= \begin{pmatrix} 1 & \Delta t \\ -\frac{k\Delta t}{m} & 1 - \frac{b\Delta t}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}^n \end{aligned}$$

For stability, we want the eigenvalues of the iteration matrix

$$\begin{pmatrix} 1 & \Delta t \\ -\frac{k\Delta t}{m} & 1 - \frac{b\Delta t}{m} \end{pmatrix}$$

to have magnitude less than one, so that the computed solution decays to zero over time.

3. The eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 + \left(\frac{b\Delta t}{m} - 2\right)\lambda + \frac{k\Delta t^2}{m} - \frac{b\Delta t}{m} + 1 = 0$$

The discriminant is

$$\left(\frac{b\Delta t}{m} - 2\right)^2 - 4\left(\frac{k\Delta t^2}{m} - \frac{b\Delta t}{m} + 1\right) = (b^2 - 4mk)\frac{\Delta t^2}{m^2}$$

Therefore when $b^2 < 4mk$ we have two complex roots, given by the formula

$$\lambda_{1,2} = 1 - \frac{b\Delta t}{2m} \pm i\frac{\Delta t}{2m}\sqrt{4km - b^2}$$

4. The magnitude of either of these eigenvalues is given by

$$\|\lambda_{1,2}\|^2 = \left(1 - \frac{b\Delta t}{2m}\right)^2 + \left(\frac{\Delta t}{2m}\sqrt{4km - b^2}\right)^2 = 1 - \frac{b\Delta t}{m} + \frac{k\Delta t^2}{m}$$

For stability we need

$$\|\lambda_{1,2}\|^2 < 1 \Leftrightarrow 1 - \frac{b\Delta t}{m} + \frac{k\Delta t^2}{m} < 1 \Leftrightarrow \frac{k\Delta t^2}{m} < \frac{b\Delta t}{m} \Leftrightarrow \Delta t < b/k$$