CS205 Homework #8

Problem 1

Give a criterion for the well-posedness of the kth order, scalar, homogeneous, constant-coefficient ODE

$$u^{(k)} + c_{k-1}u^{(k-1)} + \cdots + c_1 u' + c_0 u = 0$$

(Hint: Transform to a first-order system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and observe $\mathbf{A}$ is a matrix we've encountered previously in homework 3 problem 2)

Problem 2

Consider the system of linear ODE's

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

1. Consider the initial value problem with the above ode and the initial values
$$y_1(0) = y_2(0) = 1$$
Show that the analytic solution to this initial value problem is
$$y_1(t) = y_2(t) = e^{-t}$$

2. If we use an integration method (such as Forward/Backward Euler, or trapezoidal rule) to compute the solution to this ODE numerically, will we get the same asymptotic behavior as the analytic solution as $t \to \infty$?

Problem 3

Consider the equation of motion for a simple, damped, 1D oscillator (a zero rest length spring in 1D with damping)

$$F(x, v) = ma = -bv - kx$$

where k is the spring constant, b the (constant) damping coefficient, $v = x_t$ the velocity and $a = v_t = x_{tt}$ the acceleration.

1. Show that this 2nd order ODE is equivalent to the 1st order linear system of ODEs
$$\begin{pmatrix} x \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

2. Assume that we are using Forward Euler to solve this system numerically, with a timestep equal to Δt. If $\lambda_1, \lambda_2 \in \mathbb{C}$ are the complex eigenvalues of the matrix
$$\begin{pmatrix} 1 & \Delta t \\ -\frac{k\Delta t}{m} & 1 - \frac{b\Delta t}{m} \end{pmatrix}$$
show that the condition for stability is $\|\lambda_1\| < 1$ and $\|\lambda_2\| < 1$

3. Show that if $b^2 < 4km$ (such spring systems are referred to as *under-damped*), then the eigenvalues of the matrix above are given as
$$\lambda_{1,2} = 1 - \frac{b\Delta t}{2m} \pm i\frac{\Delta t}{2m}\sqrt{4km - b^2}$$

4. Show that if $b^2 < 4km$ the condition for stability is $\Delta t < b/k$.