CS205 Homework #7

Problem 1

We have seen the application of the conjugate gradient algorithm on the solution of symmetric, positive definite systems. Now assume that in the system $\mathbf{Ax} = \mathbf{b}$, the $n \times n$ matrix $\mathbf{A}$ is symmetric positive semi-definite with a nullspace of dimension $p < n$. This problem illustrates that one can use a modified version of conjugate gradients to solve this system as well.

1. Prove that we can write $\mathbf{A}$ as
$$\mathbf{A} = \mathbf{M}\tilde{\mathbf{A}}\mathbf{M}^T$$
where $\mathbf{M}$ is an $n \times (n-p)$ matrix with orthonormal columns that form a basis for the column space of $\mathbf{A}$, while $\tilde{\mathbf{A}}$ is an $(n-p) \times (n-p)$ symmetric *positive* definite matrix (no nullspace) [Hint: Use the diagonal form of $\mathbf{A} = \mathbf{Q\Lambda Q}^T$]

2. Let the $n \times n$ matrix $\mathbf{P}$ be defined as $\mathbf{P} = \mathbf{MM}^T$. Explain (no formal proof required) why this is a projection matrix and onto what space it projects. How can we compute $\mathbf{P}$ without knowledge of the eigenvalues-eigenvectors of $\mathbf{A}$?

3. Show that, in order to have a solution to $\mathbf{Ax} = \mathbf{b}$, we must be able to write
$$\mathbf{b} = \mathbf{M}\tilde{\mathbf{b}}$$
for an appropriate vector $\tilde{\mathbf{b}} \in \mathbb{R}^{n-p}$

4. Let $\tilde{\mathbf{x}}$ be the solution to the system $\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ and explain why $\tilde{\mathbf{x}}$ is unique. Show that any solution to the original system $\mathbf{Ax} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{M}\tilde{\mathbf{x}} + \mathbf{x}_0$ where $\mathbf{x}_0$ is in the nullspace of $\mathbf{A}$.

5. Consider the conjugate gradients algorithm for solving $\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$

$$\begin{aligned}
&\tilde{\mathbf{x}}_0 = \text{initial guess} \\
&\tilde{\mathbf{s}}_0 = \tilde{\mathbf{r}}_0 = \tilde{\mathbf{b}} - \tilde{\mathbf{A}}\tilde{\mathbf{x}}_0 \\
&\textbf{for } k = 0, 1, \ldots, 2 \\
&\quad \tilde{\alpha}_k = \frac{\tilde{\mathbf{r}}_k^T \tilde{\mathbf{r}}_k}{\tilde{\mathbf{s}}_k^T \tilde{\mathbf{A}} \tilde{\mathbf{s}}_k} \\
&\quad \tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{x}}_k + \tilde{\alpha}_k \tilde{\mathbf{s}}_k \\
&\quad \tilde{\mathbf{r}}_{k+1} = \tilde{\mathbf{r}}_k - \tilde{\alpha}_k \tilde{\mathbf{A}} \tilde{\mathbf{s}}_k \\
&\quad \tilde{\mathbf{s}}_{k+1} = \tilde{\mathbf{r}}_{k+1} + \frac{\tilde{\mathbf{r}}_{k+1}^T \tilde{\mathbf{r}}_{k+1}}{\tilde{\mathbf{r}}_k^T \tilde{\mathbf{r}}_k} \tilde{\mathbf{s}}_k \\
&\textbf{end}
\end{aligned}$$

Show that we can compute a solution to the original system $\mathbf{Ax} = \mathbf{b}$ by using the following modification of the algorithm

$$\begin{aligned}
&\mathbf{x}_0 = \text{initial guess} \\
&\mathbf{s}_0 = \mathbf{r}_0 = \mathbf{P}(\mathbf{b} - \mathbf{A}\mathbf{x}_0) \\
&\textbf{for } k = 0, 1, \ldots, 2 \\
&\quad \alpha_k = \frac{\mathbf{r}_k^T \mathbf{r}_k}{\mathbf{s}_k^T \mathbf{A} \mathbf{s}_k} \\
&\quad \mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{s}_k \\
&\quad \mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k \mathbf{PA} \mathbf{s}_k \\
&\quad \mathbf{s}_{k+1} = \mathbf{r}_{k+1} + \frac{\mathbf{r}_{k+1}^T \mathbf{r}_{k+1}}{\mathbf{r}_k^T \mathbf{r}_k} \mathbf{s}_k \\
&\textbf{end}
\end{aligned}$$

[Hint: Show that $\mathbf{x}_k = \mathbf{M}\tilde{\mathbf{x}}_k, \mathbf{r}_k = \mathbf{M}\tilde{\mathbf{r}}_k, \mathbf{s}_k = \mathbf{M}\tilde{\mathbf{s}}_k, \tilde{\alpha}_k = \alpha_k$]

Problem 2

Consider a real function $f(x)$ that is differentiable on an interval $[a, b]$.

1. Find a quadratic polynomial $g(x)$ that approximates $f(x)$ on $[a, b]$ in that $f'(a) = g'(a)$, $f'(b) = g'(b)$ and $f\left(\frac{a+b}{2}\right) = g\left(\frac{a+b}{2}\right)$ [Hint: Consider expressing $g(x)$ as a quadratic polynomial of $(x - \frac{a+b}{2})$].

2. Define a numerical quadrature rule for $\int_a^b f(x)\,dx$ by integrating the interpolant $g(x)$ on $[a, b]$.

3. Prove that this integration scheme has degree of accuracy equal to 3.

4. Define the corresponding composite quadrature rule for $\int_a^b f(x)\,dx$ we obtain by sub-dividing $[a, b]$ into the n sub-intervals $\left[a + k\frac{b-a}{n}, a + (k+1)\frac{b-a}{n}\right]$

Problem 3

The first order divided difference is given by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

When x_0 is close to x_1 we have the approximation

$$f[x_0, x_1] \approx f'\left(\frac{x_0 + x_1}{2}\right)$$

Now let $z = (x_0 + x_1)/2$, $h = (x_1 - x_0)/2$ then the error is given as

$$E = f[x_0, x_1] - f'\left(\frac{x_0 + x_1}{2}\right) = \frac{f(z+h) - f(z-h)}{2h} - f'(z)$$

Prove that the error is

$$E = \frac{h^2}{6} f'''(z) + O(h^3)$$