CS205 Homework #6 Solutions

Problem 1

1. Let $\mathbf{A}$ be a symmetric and positive definite $n \times n$ matrix. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ prove that the operation $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = \mathbf{x}^T \mathbf{A} \mathbf{y} = \mathbf{x} \cdot \mathbf{A} \mathbf{y}$ is an inner product on $\mathbb{R}^n$. That is, show that the following properties are satisfied
   (a) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}} = \langle \mathbf{u}, \mathbf{z} \rangle_{\mathbf{A}} + \langle \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}}$
   (b) $\langle \alpha \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = \alpha \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}}$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{A}}$
   (d) $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{A}} \geq 0$ and equality holds if and only if $\mathbf{u} = \mathbf{0}$
2. Which of those properties, if any, fail to hold when $\mathbf{A}$ is not positive definite? Which fail to hold if it is not symmetric?

Solution

1. (a) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}} = (\mathbf{u} + \mathbf{v})^T \mathbf{A} \mathbf{z} = \mathbf{u}^T \mathbf{A} \mathbf{z} + \mathbf{v}^T \mathbf{A} \mathbf{z} = \langle \mathbf{u}, \mathbf{z} \rangle_{\mathbf{A}} + \langle \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}}$
   (b) $\langle \alpha \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = (\alpha \mathbf{u})^T \mathbf{A} \mathbf{v} = \alpha (\mathbf{u}^T \mathbf{A} \mathbf{v}) = \alpha \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}}$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = \mathbf{u}^T \mathbf{A} \mathbf{v} = \mathbf{u}^T \mathbf{A}^T \mathbf{v} = \mathbf{A} \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{A} \mathbf{u} = \mathbf{v}^T \mathbf{A} \mathbf{u} = \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{A}}$ by symmetry
   (d) $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{A}} = \mathbf{u}^T \mathbf{A} \mathbf{u} \geq 0$ if $\mathbf{u} \neq 0$ by positive definiteness and equality holds trivially when $\mathbf{u} = 0$.
2. Property (3) holds if and only if $\mathbf{A}$ is symmetric. Property (4) holds if and only if $\mathbf{A}$ is positive definite (by definition)

Problem 2

1. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be an $\mathbf{A}$-orthogonal set of vectors, that is $\mathbf{x}_i^T \mathbf{A} \mathbf{x}_j = 0$ for $i \neq j$. Show that if $\mathbf{A}$ is symmetric and positive definite, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent. Does this hold when $\mathbf{A}$ is symmetric but not positive definite?
2. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent vectors of $\mathbb{R}^n$ and $\mathbf{A}$ a $n \times n$ symmetric positive definite matrix. Show that we can use the Gram-Schmidt algorithm to create a *full* $\mathbf{A}$-orthogonal set of n vectors. That is, subtracting from $\mathbf{x}_i$ its $\mathbf{A}$-overlap with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}$ will never create a zero vector.

Solution

1. Suppose there is some $\mathbf{x}_k$ that is the linear combination of other guys i.e.:

$$\mathbf{x}_k = \alpha_1 \mathbf{x}_{y_1} + \alpha_2 \mathbf{x}_{y_2} + \cdots + \alpha_k \mathbf{x}_{y_k}$$

If we multiply from the left by $\mathbf{x}_k^T \mathbf{A}$ we get:

$$\mathbf{x}_k^T \mathbf{A} \mathbf{x}_k = \alpha_1 \mathbf{x}_k^T \mathbf{A} \mathbf{x}_{y_1} + \alpha_2 \mathbf{x}_k^T \mathbf{A} \mathbf{x}_{y_2} + \cdots + \alpha_k \mathbf{x}_k^T \mathbf{A} \mathbf{x}_{y_k} = 0 + 0 + \cdots + 0 = 0$$

If $\mathbf{A}$ is postive definite then $\mathbf{x}_k^T \mathbf{A} \mathbf{x}_k > 0$ giving a contradiction. Note that symmetry alone is not sufficient as if $\mathbf{A} = \mathbf{0}$ then every vector is A-orthogonal to every other vector.

2. The Gram-Schmidt algorithm for $\mathbf{A}$-orthogonalization of a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is

$$\tilde{\mathbf{x}}_i = \mathbf{x}_i - \sum_{j=1}^{i-1} \frac{\mathbf{x}_i \cdot \mathbf{A} \tilde{\mathbf{x}}_j}{\tilde{\mathbf{x}}_j \cdot \mathbf{A} \tilde{\mathbf{x}}_j} \tilde{\mathbf{x}}_j$$

(With optional rescaling of the resulting vectors so that their $\mathbf{A}$-norm is equal to 1). We can see that each of the $\tilde{\mathbf{x}}_i$'s is just a linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i$, using induction. Indeed, $\tilde{\mathbf{x}}_1$ is just equal to $\mathbf{x}_1$ and $\tilde{\mathbf{x}}_i$ results from $\mathbf{x}_i$ after the subtraction of some *scalar* multiples of $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_{i-1}$. But each of them is just a linear combination of $\mathbf{x}_j$'s with $j < i$ (using the inductive hypothesis). Therefore, in each step of the algorithm, the sum $\sum_{j=1}^{i-1} \frac{\mathbf{x}_i \cdot \mathbf{A} \tilde{\mathbf{x}}_j}{\tilde{\mathbf{x}}_j \cdot \mathbf{A} \tilde{\mathbf{x}}_j} \tilde{\mathbf{x}}_j$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}$ and therefore linearly independent of $\mathbf{x}_i$. Therefore, none of the $\tilde{\mathbf{x}}_i$'s thus created can ever be equal to zero.

Problem 3

Let $\mathbf{A}$ be a $n \times n$ symmetric positive definite matrix. Consider the steepest descent method for the minimization of the function

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{b}^T \mathbf{x} + c$$

1. Let $\mathbf{x}_{\min}$ be the value that minimizes $f(\mathbf{x})$. Show that

$$f(\mathbf{x}_{\min}) = c - \frac{1}{2} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}$$

2. If $\mathbf{x}_k$ is the k-th iterate, show that

$$f(\mathbf{x}_k) - f(\mathbf{x}_{\min}) = \frac{1}{2} \mathbf{r}_k^T \mathbf{A}^{-1} \mathbf{r}_k$$

3. Show that

$$\mathbf{r}_{k+1} = \left(\mathbf{I} - \frac{\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)\mathbf{r}_k$$

4. Show that

$$[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min})] = [f(\mathbf{x}_k) - f(\mathbf{x}_{\min})]\left(1 - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k)}\right)$$

5. Show that

$$[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min})] \leq [f(\mathbf{x}_k) - f(\mathbf{x}_{\min})]\left(1 - \frac{\sigma_{\min}}{\sigma_{\max}}\right)$$

   where $\sigma_{\min}, \sigma_{\max}$ are the minimum and maximum singular values of $\mathbf{A}$, respectively.

6. What does the result of (5) imply for the convergence speed of steepest descent?

[Note: Even if you fail to prove one of (1)-(6) you may still use it to answer a subsequent question]

Solution

1. Recall that $\nabla f(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$ so $\mathbf{x}_{\min} = \mathbf{A}^{-1}\mathbf{b}$. Substituting,

$$\begin{aligned}
f(\mathbf{x}_{\min}) &= \frac{1}{2}(\mathbf{A}^{-1}\mathbf{b})^T\mathbf{A}(\mathbf{A}^{-1}\mathbf{b}) - \mathbf{b}^T(\mathbf{A}^{-1}\mathbf{b}) + c \\
&= \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{A}^{-1}\mathbf{b} - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} + c \\
&= \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} + c \\
&= c - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}
\end{aligned}$$

2. Recall that $\mathbf{r} = \mathbf{b} - \mathbf{A}\mathbf{x}$. Now proceed as:

$$\begin{aligned}
f(\mathbf{x}_k) - f(\mathbf{x}_{\min}) &= \frac{1}{2}\mathbf{x}_k^T\mathbf{A}\mathbf{x}_k - \mathbf{b}^T\mathbf{x}_k + c - c + \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \\
&= \frac{1}{2}\mathbf{x}_k^T\mathbf{A}\mathbf{x}_k - \mathbf{b}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{x}_k + \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \\
&= \frac{1}{2}\mathbf{x}_k^T\mathbf{A}\mathbf{x}_k - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{x}_k - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{A}\mathbf{x}_k + \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} \\
&= \frac{1}{2}(\mathbf{A}\mathbf{x}_k - \mathbf{b})^T\mathbf{x}_k - \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}_k - \mathbf{b}) \\
&= -\frac{1}{2}\mathbf{r}_k^T\mathbf{x}_k + \frac{1}{2}\mathbf{b}^T\mathbf{A}^{-1}\mathbf{r}_k
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{A}\mathbf{x}_k + \frac{1}{2}\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{b} \\
&= -\frac{1}{2}\mathbf{r}_k^T\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}_k - \mathbf{b}) \\
&= \frac{1}{2}\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r_k}
\end{aligned}
$$

3.

$$
\begin{aligned}
\mathbf{r}_{k+1} &= \mathbf{r}_k - \alpha_k\mathbf{A}\mathbf{r}_k \\
&= \mathbf{r}_k - \mathbf{A}\mathbf{r}_k\frac{\mathbf{r}_k^T\mathbf{r}_k}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k} \\
&= \left(\mathbf{I} - \frac{\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)\mathbf{r}_k
\end{aligned}
$$

4.

$$
\begin{aligned}
f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min}) &= \frac{1}{2}\mathbf{r}_{k+1}^T\mathbf{A}^{-1}\mathbf{r_{k+1}} \\
&= \frac{1}{2}\mathbf{r}_k^T\left(\mathbf{I} - \frac{\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)^T\mathbf{A}^{-1}\left(\mathbf{I} - \frac{\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)\mathbf{r_k} \\
&= \frac{1}{2}\mathbf{r}_k^T\left(\mathbf{A}^{-1} - 2\frac{\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k} + \frac{\mathbf{r}_k\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)^2}\right)\mathbf{r_k} \\
&= \frac{1}{2}\mathbf{r}_k^T\left(\mathbf{A}^{-1} - 2\frac{\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k} + \frac{\mathbf{r}_k(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)\mathbf{r}_k^T}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)^2}\right)\mathbf{r_k} \\
&= \frac{1}{2}\mathbf{r}_k^T\left(\mathbf{A}^{-1} - \frac{\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)\mathbf{r_k} \\
&= \frac{1}{2}\left(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right) \\
&= \frac{1}{2}\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k\left(1 - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k)}\right) \\
&= (f(\mathbf{x}_k) - f(\mathbf{x}_{\min}))\left(1 - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k)}\right)
\end{aligned}
$$

5. In the review session we proved that

$$
\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k \le \sigma_{\max}^{\mathbf{A}}\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k
$$
$$
\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k \le \sigma_{\max}^{\mathbf{A}^{-1}}\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k = \tfrac{1}{\sigma_{\min}^{\mathbf{A}}}\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k
$$

Therefore

$$
(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k) \le \frac{\sigma_{\max}}{\sigma_{\min}}(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)^2
$$

or

$$\frac{(\mathbf{r}_k^T \mathbf{r}_k)^2}{(\mathbf{r}_k^T \mathbf{A} \mathbf{r}_k)(\mathbf{r}_k^T \mathbf{A}^{-1} \mathbf{r}_k)} \geq \frac{\sigma_{\min}}{\sigma_{\max}}$$

Thus, using (4)

$$[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min})] \leq [f(\mathbf{x}_k) - f(\mathbf{x}_{\min})] \left(1 - \frac{\sigma_{\min}}{\sigma_{\max}}\right)$$

6. This result shows that the speed of convergence is associated with the condition number of **A**. With a perfectly conditioned matrix (which has to be a multiple of the identity, if it is symmetric) steepest descent will converge in 1 step. In a matrix with a condition number equal to κ, in each step of steepest descent, the distance of the current function value from the minimum value will shrink by a factor of $1 - 1/\kappa$.