CS205 Homework #6

Problem 1

1. Let $\mathbf{A}$ be a symmetric and positive definite $n \times n$ matrix. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ prove that the operation $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}} = \mathbf{x}^T \mathbf{A} \mathbf{y} = \mathbf{x} \cdot \mathbf{A} \mathbf{y}$ is an inner product on $\mathbb{R}^n$. That is, show that the following properties are satisfied
   (a) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}} = \langle \mathbf{u}, \mathbf{z} \rangle_{\mathbf{A}} + \langle \mathbf{v}, \mathbf{z} \rangle_{\mathbf{A}}$
   (b) $\langle \alpha \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = \alpha \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}}$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} = \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{A}}$
   (d) $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{A}} \geq 0$ and equality holds if and only if $\mathbf{u} = \mathbf{0}$
2. Which of those properties, if any, fail to hold when $\mathbf{A}$ is not positive definite? Which fail to hold if it is not symmetric?

Problem 2

1. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be an $\mathbf{A}$-orthogonal set of vectors, that is $\mathbf{x}_i^T \mathbf{A} \mathbf{x}_j = 0$ for $i \neq j$. Show that if $\mathbf{A}$ is symmetric and positive definite, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent. Does this hold when $\mathbf{A}$ is symmetric but not positive definite?
2. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent vectors of $\mathbb{R}^n$ and $\mathbf{A}$ a $n \times n$ symmetric positive definite matrix. Show that we can use the Gram-Schmidt algorithm to create a *full* $\mathbf{A}$-orthogonal set of n vectors. That is, subtracting from $\mathbf{x}_i$ its $\mathbf{A}$-overlap with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}$ will never create a zero vector.

Problem 3

Let $\mathbf{A}$ be a $n \times n$ symmetric positive definite matrix. Consider the steepest descent method for the minimization of the function

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{b}^T \mathbf{x} + c$$

1. Let $\mathbf{x}_{\min}$ be the value that minimizes $f(\mathbf{x})$. Show that

$$f(\mathbf{x}_{\min}) = c - \frac{1}{2} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}$$

2. If $\mathbf{x}_k$ is the k-th iterate, show that

$$f(\mathbf{x}_k) - f(\mathbf{x}_{\min}) = \frac{1}{2} \mathbf{r}_k^T \mathbf{A}^{-1} \mathbf{r}_k$$

3. Show that

$$\mathbf{r}_{k+1} = \left(\mathbf{I} - \frac{\mathbf{A}\mathbf{r}_k\mathbf{r}_k^T}{\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k}\right)\mathbf{r}_k$$

4. Show that

$$[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min})] = [f(\mathbf{x}_k) - f(\mathbf{x}_{\min})]\left(1 - \frac{(\mathbf{r}_k^T\mathbf{r}_k)^2}{(\mathbf{r}_k^T\mathbf{A}\mathbf{r}_k)(\mathbf{r}_k^T\mathbf{A}^{-1}\mathbf{r}_k)}\right)$$

5. Show that

$$[f(\mathbf{x}_{k+1}) - f(\mathbf{x}_{\min})] \leq [f(\mathbf{x}_k) - f(\mathbf{x}_{\min})]\left(1 - \frac{\sigma_{\min}}{\sigma_{\max}}\right)$$

   where $\sigma_{\min}, \sigma_{\max}$ are the minimum and maximum singular values of $\mathbf{A}$, respectively.

6. What does the result of (5) imply for the convergence speed of steepest descent?

[Note: Even if you fail to prove one of (1)-(6) you may still use it to answer a subsequent question]