CS205 Homework #5 Solutions

Problem 1

[Heath 5.5, p.248]

1. Show that the iterative method
$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$
is mathematically equivalent to the secant method for solving a scalar nonlinear equation $f(x) = 0$.

2. When implemented in finite-precision floating-point arithmetic, what advantages or disadvantages does the formula given in part (1) have compared with the formula for the secant method (given in the notes and in Heath, section 5.5.4)?

Solutions

1. Starting with the secant method update formula is given as
$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \\ &= \frac{x_k(f(x_k) - f(x_{k-1})) - f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \\ &= \frac{x_k f(x_k) - x_k f(x_{k-1}) - x_k f(x_k) + x_{k-1}f(x_k)}{f(x_k) - f(x_{k-1})} \\ &= \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} \end{aligned}$$

2. When we're near the solution x_{k-1} and x_k are close to each other and thus their difference is near zero. Similarly $f(x_{k-1})$ and $f(x_k)$ are close to each other so their difference is near zero. Thus we have an indefinite form in secant computing $(x_k - x_{k-1})/(f(x_k) - f(x_{k-1}))$. However, since we're near the root we have $f(x_k)$ near zero so it cancels the term and we're left with $x_{k+1} = x_k$. Thus, as our indefiniteness increases finite precision gives us a fixed point. However, in the form given in this problem, the indefinite division is the only thing affecting x_{k+1}.

Problem 2

[Heath 5.6, p.249] Suppose we wish to develop an iterative method to compute the square root of a given positive number y, i.e., to solve the nonlinear equation $f(x) = x^2 - y = 0$

given the value of y. Each of the functions g_1 and g_2 listed next gives a fixed-point problem that is equivalent to the equation $f(x) = 0$. For each of these functions, determine whether the corresponding fixed-point iteration scheme $x_{k+1} = g_i(x_k)$ is locally convergent to $\sqrt{y}$ if $y = 3$. Explain your reasoning in each case.

1. $g_1(x) = y + x - x^2$.
2. $g_2(x) = 1 + x - x^2/y$.
3. What is the fixed-point iteration *function* given by Newton's method for this particular problem?

Solutions

$x^* = \sqrt{3}$. Thus we need to determine whether each iteration is locally convergent or devergent by examining $|g'(x^*)|$. We have

1. $g_1'(x) = 1 - 2x$ so $|g_1'(x^*)| = |2\sqrt{3} - 1| > |2 - 1| = 1$. Thus divergent
2. $g_2'(x) = 1 - 2x/y$ so $|g_2'(x^*)| = |2\sqrt{3}/3 - 1| < |4/3 - 1| < 1$ so locally convergent.
3. We have $f'(x) = 2x$ so $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - y}{2x_k}$ but if $y = 3$ we get

$$x_{k+1} = x_k - \frac{x_k^2 - 3}{2x_k} = \frac{x_k}{2} + \frac{3}{2x_k}$$

Problem 3

[Heath 5.11, p.249] Suppose you are using the secant method to find a root x^* of a nonlinear equation $f(x) = 0$. Show that if at any iteration it happens to be the case that either $x_k = x^*$ or $x_{k-1} = x^*$ (but not both), then it will also be true that $x_{k+1} = x^*$.

Solution

Recall the secant update is given as $x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$ Now substitute $x_k = x^*$ then:

$$x_{k+1} = x^* - \frac{f(x^*)(x^* - x_{k-1})}{f(x^*) - f(x_{k-1})} = x^* - \frac{0(x^* - x_{k-1})}{0 - f(x_{k-1})} = x^*.$$

Now substitute $x_{k-1} = x^*$ then:

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x^*)}{f(x_k) - f(x^*)} = x_k - \frac{f(x_k)(x_k - x^*)}{f(x_k)} = x_k - (x_k - x^*) = x^*.$$

Problem 4

[Heath 6.8, p.302] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \frac{1}{2}(x_1^2 - x_2)^2 + \frac{1}{2}(1 - x_1)^2$$

1. At what point does f attain a minimum?
2. Perform one iteration of Newton's method for minimizing f using as starting point $\mathbf{x}_0 = \begin{bmatrix} 2 & 2 \end{bmatrix}^T$
3. In what sense is this a good step?
4. In what sense is this a bad step?

[Note: The Newton method for optimization of a function of multiple variables $f : \mathbb{R}^n \to \mathbb{R}$ gives the update step $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ as the solution to the system $\mathbf{H}(\mathbf{x}_k)\mathbf{s}_k = -\nabla f(\mathbf{x}_k)$, where $\mathbf{H}(\mathbf{x}_k)$ is the Hessian matrix of f evaluated at $\mathbf{x}_k$. See also Heath, section 6.5.3.]

Solution

1. To find where $f(\mathbf{x})$ obtains a minimum we just solve $\nabla f(\mathbf{x}) = \mathbf{o}$. So we have $\nabla f(\mathbf{x}) = (2x_1^3 - 2x_1x_2 + x_1 - 1, x_2 - x_1^2) = (0, 0)$. The second equation implies $x_2 = x_1^2$ so substituting into the first we get $2x_1^3 - 2x_1^3 + x_1 - 1 = 0$ so we get $x_1 = 1$. Then from the second we get $x_2 = 1$. Thus the min is at $\mathbf{x} = (1, 1)$.

2. The Hessian matrix is given has

$$\mathbf{H}_f(\mathbf{x}) = \begin{bmatrix} 6x_1^2 - 2x_2 + 1 & -2x_1 \\ -2x_1 & 1 \end{bmatrix}.$$

   We need $\mathbf{H}(\mathbf{x}_k)\mathbf{s}_k = -\nabla f(\mathbf{x}_k)$, plugging in we get

$$\begin{bmatrix} 21 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$$

   Inverting the matrix we get $H^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 21 & 4 \end{bmatrix}$ and so

$$\Delta\mathbf{x}_2 = \begin{bmatrix} 1 & 4 \\ 21 & 4 \end{bmatrix} \begin{bmatrix} -9 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

   So we get

$$\mathbf{x}_2 = \Delta\mathbf{x}_2 + \mathbf{x}_1 = \begin{bmatrix} 9/5 \\ 16/5 \end{bmatrix}$$

3. In what sense is this a good step?

   The step is good in that the derivative is closer to zero. i.e. $\|f'(\mathbf{x}_0)\| = \|f'(2,2)\|^2 = \|(9,-2)\| \approx 9.2$ and $\|f'(\mathbf{x}_1)\| = \|f'(9/5, 16/5)\| \approx 0.94$

4. In what sense is this a bad step?

   $\|\mathbf{x}_0 - \mathbf{x}^*\| = \|(1,1)\| \approx 1.41$. $\|\mathbf{x}_1 - \mathbf{x}^*\| = \|(6/5, 4/5)\| \approx 2.34$. So the new iterate is further away from the solution.