CS 205a Fall 2010 Midterm 1

Please write your name on the top right of the first page. The exam is closed book, and no calculators are allowed. You have 1 hour and 15 minutes to complete the exam.

Multiple Choice (1 pt each)

For each of the following questions, circle all answers which are correct. You must circle all of the answers for a given question correctly to receive credit.

1. Which of the following can be said about a Householder matrix $H = I - 2\frac{vv^T}{v^Tv}$
   - (a) The condition number for H is 1
   - (b) It is a projection matrix onto the hyperplane orthogonal to v
   - (c) It preserves the 2-norm of a vector (ie $\|x\|_2 = \|Hx\|_2$ for all x)
   - (d) One of the eigenvalues of H is 1 with multiplicity $n-1$
2. Which of the following classes of matrices are positive semi-definite?
   - (a) Permutation matrices
   - (b) Projection matrices
   - (c) Reflection matrices
   - (d) Symmetric matrices
3. Suppose that a square matrix A is ill-conditioned. Which of the following matrices could potentially have a better condition number?
   - (a) cA, where c is a non-zero scalar
   - (b) DA, where D is a nonsingular diagonal matrix
   - (c) PA, where P is a permutation matrix
   - (d) A^{-1}
4. Which of the following about the least squares solutions are true?
   - (a) It satisfies $Ax = b$
   - (b) It can be found by solving the normal equation
   - (c) Its associated residual lies in the nullspace of A^T
   - (d) It lies in the column space of A

Eigenvalues (10 pts)

1. Given that $A = T^{-1}BT$, prove that A and B have the same eigenvalues. How do the eigenvectors of A and B relate? (2 pts)

2. What are the eigenvalues of a projection matrix? (Just state. No proof required) (1 pts)

3. If $A = A^T$, and $x^T A x > 0, \forall x \neq 0$, prove that the eigenvalues of A are all positive. (3 pts)

4. Given a symmetric matrix, how might you find the second largest eigenvalue. (4 pts)

SVD (2 pts)

1. State (without proof) SVD of the following matrices: (1 pt each)

   (a)

$$\begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

   (b)

$$\begin{pmatrix} 0 & 0.0637 & 0 \\ 97.83 & 0 & 0 \\ 0 & 0 & 2.763 \end{pmatrix}$$

Least Squares (6 pts)

The complete orthogonal factorization of matrix A is $A = QRZ^T$, where $R = \begin{pmatrix} \hat{R} & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{R}$ is upper triangular and non-singular, and Q, Z are orthogonal. Given the complete orthogonal factorization of A, construct an algorithm for computing the minimum norm least squares solution of the problem $Ax = b$. (Note that the QR factorization is not sufficient, and the complete orthogonal factorization is cheaper than the SVD.)

Nonlinear equations (12 pts)

The Taylor series for a sufficiently smooth function $f(x)$ is given by

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n,$$

where $f^{(n)}(x)$ is the n^{th} derivative of f evaluation at x. Truncating the series to the first k terms gives

$$f(x+h) = \sum_{n=0}^{k-1} \frac{f^{(n)}(x)}{n!} h^n + R_k(x),$$

where $R_k(x)$ is the error due to truncation. The Taylor theorem says that there exists a $\xi \in [x, x+h]$ s.t.

$$R_k(x) = \frac{f^{(k)}(\xi)}{k!} h^k$$

1. Derive Newton's method for finding the root of $f(x)$ by truncating the Taylor series. (2 pts)

2. Derive an update rule for finding the root of $f(x)$ by truncating the series to the first 3 terms. Note that the update rule should provide a unique guess for the next iteration. (4 pts)

3. Prove that Newton's method converges at a quadratic rate. What additional assumptions are required? Note that an iterative method is said to converge with rate r if $\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^r} = C$ for some finite constanct $C > 0$, whre e_k is the error at iteration k. (Hint: Use taylor series expansion on $g(x) = \frac{f(x)}{f'(x)}$ around the root x^* of f.) (6 pts)

Optimization (6 pts)

1. The steepest descent method involves choosing a search direction $\vec{s_k} = -\nabla f(\vec{x_k})$, choosing α_k to minimize $f(\vec{x_k} + \alpha_k \vec{s_k})$ and setting $\vec{x_{k+1}} = \vec{x_k} + \alpha_k \vec{s_k}$. Prove that $\nabla f(\vec{x_{k+1}}) \cdot \nabla f(\vec{x_k}) = 0$.