CS205 Homework #3

Problem 1

Consider an $n \times n$ matrix $\mathbf{A}$.

1. Show that if $\mathbf{A}$ has distinct eigenvalues all the corresponding eigenvectors are linearly independent.

2. Show that if $\mathbf{A}$ has a full set of eigenvectors (i.e. any eigenvalue λ with multiplicity k has k corresponding linearly independent eigenvectors), it can be written as $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix of $\mathbf{A}$'s eigenvalues and $\mathbf{Q}$'s columns are $\mathbf{A}$'s eigenvectors. Hint: show that $\mathbf{AQ} = \mathbf{Q\Lambda}$ and that $\mathbf{Q}$ is invertible.

3. If $\mathbf{A}$ is symmetric show that any two eigenvectors corresponding to different eigenvalues are orthogonal.

4. If $\mathbf{A}$ is symmetric show that it has a full set of eigenvectors. Hint: If $(\lambda, \mathbf{q})$ is an eigenvalue, eigenvector ($\mathbf{q}$ normalized) pair and λ is of multiplicity $k > 1$, show that $\mathbf{A} - \lambda\mathbf{q}\mathbf{q}^T$ has an eigenvalue of λ with multiplicity $k - 1$. To show that consider the Householder matrix $\mathbf{H}$ such that $\mathbf{Hq} = \mathbf{e}_1$ and note that $\mathbf{HAH}^{-1} = \mathbf{HAH}$ and $\mathbf{A}$ are similar.

5. If $\mathbf{A}$ is symmetric show that it can be written as $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$ for an orthogonal matrix $\mathbf{Q}$. (You may use the result of (4) even if you didn't prove it)

Problem 2

[Adapted from Heath 4.23] Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Recall that these are the roots of the characteristic polynomial of $\mathbf{A}$, defined as $f(\lambda) \equiv \det(\mathbf{A} - \lambda\mathbf{I})$. Also we define the multiplicity of an eigenvalue to be the degree of it as a root of the characteristic polynomial.

1. Show that the determinant of $\mathbf{A}$ is equal to the product of its eigenvalues, i.e. $\det(\mathbf{A}) = \prod_{j=1}^{n} \lambda_j$.

2. The *trace* of a matrix is defined to be the sum of its diagonal entries, i.e., $\text{trace}(\mathbf{A}) = \sum_{j=1}^{n} a_{jj}$. Show that the trace of $\mathbf{A}$ is equal to the sum of its eigenvalues, i.e. $\text{trace}(\mathbf{A}) = \sum_{j=1}^{n} \lambda_j$.

3. Recall a matrix $\mathbf{B}$ is similar to $\mathbf{A}$ if $\mathbf{B} = \mathbf{T}^{-1}\mathbf{AT}$ for a non-singular matrix $\mathbf{T}$. Show that two similar matrices have the same trace and determinant.

4. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -\frac{a_{m-1}}{a_m} & -\frac{a_{m-2}}{a_m} & \cdots & -\frac{a_1}{a_m} & -\frac{a_0}{a_m} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

What is $\mathbf{A}$'s characteristic polynomial? Describe how you can use the power method find the largest root (in magnitude) of an arbitrary polynomial.

Problem 3

Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{A} = \mathbf{U\Sigma V}^T$ its singular value decomposition.

1. Show that $\|\mathbf{A}\|_2 = \|\mathbf{\Sigma}\|_2$
2. Show that if $m \geq n$ then for all $\mathbf{x} \in R^n$

$$\sigma_{\min} \leq \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} \leq \sigma_{\max}$$

   where $\sigma_{\min}, \sigma_{\max}$ are the smallest and largest singular values, respectively.

3. The Frobenius norm of a matrix $\mathbf{A}$ is defined as $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$. Show that $\|A\|_F = \sqrt{\sum_{i=1}^p \sigma_i^2}$ where $p = \min\{m, n\}$.
4. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an m-vector prove that the solution $\mathbf{x}$ of minimum Euclidean norm to the least squares problem $\mathbf{Ax} \cong \mathbf{b}$ is given by

$$\mathbf{x} = \sum_{\sigma_i \neq 0} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

   where $\mathbf{u}_i$, $\mathbf{v}_i$ are the columns of $\mathbf{U}$ and $\mathbf{V}$, respectively.

5. Show that the columns of $\mathbf{U}$ corresponding to non-zero singular values form an orthogonal basis of the column space of $\mathbf{A}$. What space do the columns of $\mathbf{U}$ corresponding to zero singular values span?