CS205 Homework #2

Problem 1

[Heath 3.29, page 152] Let $\mathbf{v}$ be a nonzero n-vector. The hyperplane normal to $\mathbf{v}$ is the (n-1)-dimensional subspace of all vectors $\mathbf{z}$ such that $\mathbf{v^T z} = \mathbf{0}$. A *reflector* is a linear transformation $\mathbf{R}$ such that $\mathbf{Rx} = -\mathbf{x}$ if $\mathbf{x}$ is a scalar multiple of $\mathbf{v}$, and $\mathbf{Rx} = \mathbf{x}$ if $\mathbf{v^T x} = \mathbf{0}$. Thus, the hyperplane acts as a mirror: for any vector, its component within the hyperplane is invariant, whereas its component orthogonal to the hyperplane is reversed.

1. Show that $\mathbf{R} = \mathbf{2P} - \mathbf{I}$, where $\mathbf{P}$ is the orthogonal projector onto the hyperplane normal to $\mathbf{v}$. Draw a picture to illustrate this result
2. Show that $\mathbf{R}$ is symmetric and orthogonal
3. Show that the Householder transformation

$$\mathbf{H} = \mathbf{I} - \mathbf{2}\frac{\mathbf{vv^T}}{\mathbf{v^T v}},$$

   is a reflector

4. Show that for any two vectors $\mathbf{s}$ and $\mathbf{t}$ such that $\mathbf{s} \neq \mathbf{t}$ and $\|\mathbf{s}\|_\mathbf{2} = \|\mathbf{t}\|_\mathbf{2}$, there is a reflector $\mathbf{R}$ such that $\mathbf{Rs} = \mathbf{t}$

Problem 2

Let $\mathbf{A}$ be a rectangular $m \times n$ matrix with full column rank and $m > n$. Consider the $\mathbf{QR}$ decomposition of $\mathbf{A}$.

1. Show that $\mathbf{P_0} = \mathbf{I} - \mathbf{QQ^T}$ is the projection matrix onto the nullspace of $\mathbf{A^T}$
2. Show that for every $\mathbf{x}$ we have $\|\mathbf{Ax} - \mathbf{b}\|_\mathbf{2}^\mathbf{2} = \|\mathbf{A}(\mathbf{x} - \mathbf{x_0})\|_\mathbf{2}^\mathbf{2} + \|\mathbf{Ax_0} - \mathbf{b}\|_\mathbf{2}^\mathbf{2}$ where $\mathbf{x_0}$ is the least squares solution of $\mathbf{Ax} = \mathbf{b}$
3. Show that the minimum value for the 2-norm of the residual is attained when $\mathbf{x}$ is equal to the least squares solution and that this minimum value is equal to $\|\mathbf{P_0 b}\|_\mathbf{2}$

Problem 3

State whether the following classes of matrices are positive (semi-)definite, negative (semi-)definite, indefinite, or whether their definiteness cannot be determined in general

1. Orthogonal matrices

2. Matrices of the form $\mathbf{A^T A}$ where $\mathbf{A}$ is a rectangular matrix

3. Projection matrices

4. Matrices of the form $\mathbf{I} - \mathbf{P}$ where $\mathbf{P}$ is a projection matrix

5. Householder matrices

6. Upper triangular matrices with positive diagonal elements

7. A diagonally dominant matrix with positive elements on the diagonal. A matrix is called diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ and $|a_{ii}| > \sum_{j \neq i} |a_{ji}|$.

Problem 4

[Heath 3.12 page 150]

1. Let $\mathbf{A}$ be a $n \times n$ matrix. Show that any two of the following conditions imply the other.
   (a) $\mathbf{A^T} = \mathbf{A}$
   (b) $\mathbf{A^T A} = \mathbf{I}$
   (c) $\mathbf{A^2} = \mathbf{I}$

2. Give a specific example, other than the identity matrix $\mathbf{I}$ or a permutation of it, of a 3×3 matrix that has all three of these properties.

3. Name a nontrivial class of matrices that have all three of these properties.

Problem 5

[Heath 3.16 page 150] Consider the vector $\mathbf{a}$ as an $n \times 1$ matrix.

1. Write out its QR factorization, showing the matrices $\mathbf{Q}$ and $\mathbf{R}$ explicitly.

2. What is the solution to the linear least squared problem $\mathbf{ax} \cong \mathbf{b}$, where $\mathbf{b}$ is a given n-vector?